Contribution to the Theory of Attraction when the Force Varies as any Power of the Distance.

By Major P. A. MacMahon, F.R.S., and H. B. C. Darling.

(Received April 8, 1918,—Revised December, 1918.)

Introduction.

The object of this paper is to make a contribution to the theory of attraction when the force is proportional to any given power of the distance. In the case of the law of nature, it is well known that when the attracting mass is a hollow shell of uniform density, whose exterior and interior bounding surfaces are both surfaces of revolution about the axis of z, the potential at any point exterior to the exterior surface can be simply obtained from the potential at any point on the axis exterior to the exterior bounding surface, and a similar theory connects the potential at any point within the inner bounding surface with that at any point on the axis which is also within the inner surface. In the expression for the point z on the z axis we have merely to write r for z, and intercalate the zonal harmonics P_0, P_1, P_2, ... in the well-known manner.

We consider the analogous theory when the law is that of the nth power of the distance instead of that of the inverse square.

In a notation which we find the most convenient for the expression of our results we consider the expansion

$$(1-2\mu\alpha+\alpha^2)^{\frac{1}{2}(2s-1)} = \Sigma P_{m,s}\alpha^m;$$

so that when the second suffix of P is zero the coefficients are zonal harmonics

$$P_{0,0}, \quad P_{1,0}, \quad P_{2,0} \ldots\ldots$$

instead of

$$P_0, \quad P_1, \quad P_2, \ldots\ldots$$

in the ordinary notation.

These double-suffixed coefficients have been considered by Heine and other investigators,* but not in the present notation or with the present object in view.

In the result we find that the expressions to be intercalated are linear functions of these double-suffixed coefficients, the coefficients which connect them being derived from potential functions in regard to a point on the z axis for different values of s.

* Heine, 'Kugelfunctionen,' Band I, p. 297.

The differential equation satisfied by $P_{m,s}$ is

$$(1-\mu^2)\partial_\mu{}^2 P_{m,s}+(2s-2)\mu\partial_\mu P_{m,s}+m(m-2s+1)P_{m,s}=0.$$

Integrating this equation with respect to μ in the case $m=2s$, we obtain

$$(1-\mu^2)\partial_\mu P_{2s,s}+2s\mu P_{2s,s}=\text{constant}. \tag{1}$$

When s is an integer, the left-hand side of (1) vanishes for $\mu=0$; thus the constant in (1) is zero. When s is fractional, we may suppose $P_{2s,s}$ subject to the condition that the constant in (1) is zero. Hence, from (1), we obtain

$$P_{2s,s}=C(1-\mu^2)^s,$$

where C is a constant with respect to μ. Considering the expansion of

$$(1-2\mu\alpha+\alpha^2)^{s-\frac{1}{2}}$$

when $\mu=0$, we see that $C=\dfrac{1.3.5......(2s-1)}{2.4.6......2s}$, and therefore

$$P_{2s,s}=\frac{1.3.5......(2s-1)}{2.4.6......2s}(1-\mu^2)^s. \tag{2}$$

Now put
$$P_{m+2s,s}=P_{2s,s}T_{m,s}$$

and substitute in the differential equation satisfied by $P_{m+2s,s}$. We find on reduction

$$(1-\mu^2)\partial_\mu{}^2 T_{m,s}-(2s+2)\mu\partial_\mu T_{m,s}+m(m+2s+1)T_{m,s}=0,$$

which is seen to be the same as the equation satisfied by $P_{m,-s}$. Hence since $T_{m,s}$ and $P_{m,-s}$ are of the same degree in μ they can only differ by a numerical factor.

Thus
$$P_{m+2s,s}=A\,.\,P_{2s,s}P_{m,-s},$$

where A is a numerical factor to be determined.

Putting $\mu=0$ in this equality we find that when m is uneven both sides vanish, but when m is even

$$\frac{(2s-1)(2s-3)......(-m+1)}{2.4......(m+2s)}$$
$$=A\frac{1.3...(2s-1)}{2.4...2s}\,.\,(-)^{\frac{1}{2}m}\frac{(2s+1)(2s+3)......(2s+m-1)}{2.4......m};$$

and on reduction

$$1=A\binom{m+2s}{m}.$$

Thence
$$\binom{m+2s}{m}P_{m+2s,s}=P_{2s,s}P_{m,-s}, \tag{3}$$

and therefore by (2) we see that if $n>2s-1$, then

$$P_{n,s}=0,\text{ for }\mu=1.$$

A knowledge of this property is necessary in what follows. The relation (3) may be deduced from results already published which have been otherwise obtained by different authors.

1. In the application of zonal harmonics to physical questions it is usual to deal in the first instance with surfaces and solids of revolution symmetrical about an axis. In the theory of attraction according to the law of nature we have in such a case an easy way of finding the potential at any point which is not inside the attracting material when once the potential at any point upon the axis of symmetry is known. As usually stated:—

Suppose that an attracting mass, M, of uniform density, is a hollow shell whose exterior and interior bounding surfaces are both surfaces of revolution, their common axis being the axis of z; let the origin be taken within the interior bounding surface, and let R be the distance from any point of the mass to a point upon the axis of z at a distance z from the origin and within the inner surface; then if we write the potential at the point upon the axis of z

$$\Sigma R^{-1} dM = M\Sigma a_{n,0} z^n$$

the potential at another point, not upon the z axis but within the inner surface whose polar co-ordinates are r, θ may be expressed by

$$M\Sigma a_{n,0} P_n(\cos\theta) r^n.$$

And there is a similar theorem with regard to points which are without the outer surface.

The problem before us is the determination of the analogous result when the potential is of the form

$$\Sigma R^{2s-1} dM$$

where s may be any real numerical magnitude.

The result is found to involve as coefficient of r^n a linear function of the coefficients $P_{m,s}$.

2. The effect of Laplace's operator

$$\nabla^2$$

upon the product $P_{m,s} r^n$ has been given by other investigators in a different notation.

The formula in the present notation is

$$\nabla^2 P_{m,s} r^n = \{(n-m+2s)(n+m-2s+1) P_{m,s} + 2s(2s-1) P_{m,s-1}\} r^{n-2}$$

leading to the particular case

$$\nabla^2 P_{n,s} r^n = 2s(2s-1) P_{n-2,s-1} r^{n-2}.$$

3. We first require the following Lemma:—

"The value of the integral

$$\int_0^{2\pi} \mathrm{P}_{n,s}\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\}\,d\phi$$

may be always expressed in the form

$$c_0\mathrm{P}_{n,0}(\mu) + c_1\mathrm{P}_{n,1}(\mu) + c_2\mathrm{P}_{n,2}(\mu) + \dots$$

where $\mu = \cos\theta$; $c_0, c_1, c_2, \dots$, are independent of μ; and the series is carried to $\frac{1}{2}(n+1)$ or $\frac{1}{2}(n+2)$ terms."

For, since all uneven powers of

$$\sin\theta\sin\theta'\cos(\phi-\phi')$$

will disappear on integration, we see that the integral will be an expression of the nth degree in μ, the successive powers of μ being $n, n-2, n-4, \dots$. Moreover, this is also the case with the series

$$c_0\mathrm{P}_{n,0}(\mu) + c_1\mathrm{P}_{n,1}(\mu) + \dots$$

and the unknown quantities $c_0, c_1, c_2, \dots$ are just sufficient to determine the equality without ambiguity. Further, the functions P which occur in the series will all be of the form $\mathrm{P}_{n,t}$ where $n > 2t-1$, and will therefore all vanish when $\mu = 1$, except for $t = 0$.

Hence when $\mu = 1$ the series vanishes with the exception of the first term, which is

$$c_0\mathrm{P}_{n,0}(\mu) = c_0.$$

We now write

$$r^n \int_0^{2\pi} \mathrm{P}_{n,s}\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\}\,d\phi'$$
$$= r^n\{c_0\mathrm{P}_{n,0}(\mu) + c_1\mathrm{P}_{n,1}(\mu) + c_2\mathrm{P}_{n,2}(\mu) + \dots\} \quad (1)$$

and putting $\mu = 1$, we have

$$\int_0^{2\pi} \mathrm{P}_{n,s}(\cos\theta')\,d\phi' = c_0$$

or

$$c_0 = 2\pi\mathrm{P}_{n,s}(\mu') \text{ where } \mu' = \cos\theta'.$$

Now applying Laplace's operator

$$2s(2s-1)r^{n-2}\int_0^{2\pi} \mathrm{P}_{n-2,s-1}\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\}\,d\phi'$$
$$= r^{n-2}\{1\,.\,2\,c_1\mathrm{P}_{n-2,0}(\mu) + 3\,.\,4\,c_2\mathrm{P}_{n-2,1}(\mu) + 5\,.\,6\,c_3\mathrm{P}_{n-2,2}(\mu) + \dots\}$$

and herein putting $\mu = 1$, we find that

$$c_1 = \binom{2s}{2}\,.\,2\pi\mathrm{P}_{n-2,s-1}(\mu').$$

Again applying Laplace's operator we find

$$c_2 = \binom{2s}{4} \cdot 2\pi \mathrm{P}_{n-4,\, s-2}(\mu')$$

and generally

$$c_p = \binom{2s}{2p} \cdot 2\pi \mathrm{P}_{n-2p,\, s-p}(\mu'),$$

where we must remember that $\binom{2s}{2p}$ is an abbreviated notation for an expression in which s may be any real number.

Hence putting

$$\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') = \cos\Theta$$

we see that

$$\int_0^{2\pi} \mathrm{P}_{n,s}(\cos\Theta)\, d\phi'$$

$$= 2\pi\left\{\mathrm{P}_{n,s}(\mu')\,\mathrm{P}_{n,0}(\mu) + \binom{2s}{2}\mathrm{P}_{n-2,\, s-1}(\mu')\,\mathrm{P}_{n,1}(\mu) + \ldots\ldots \right.$$
$$\left. + \binom{2s}{2p}\mathrm{P}_{n-2p,\, s-p}(\mu')\,\mathrm{P}_{n,p}(\mu) + \ldots\right\}, \quad (2)$$

an expression in which μ, μ' may be interchanged.

4. Now in the case of any solid or surface of revolution, the potential, which we will write V_s, may be expressed respectively in the forms

$$\iiint_0^{2\pi} f(r', \theta')(r^2 - 2rr'\cos\Theta + r'^2)^{s-\frac{1}{2}}\, d\phi'\, d\theta'\, dr' \quad (3)$$

and

$$\iint_0^{2\pi} f(r', \theta')(r^2 - 2rr'\cos\Theta + r'^2)^{s-\frac{1}{2}}\, d\phi'\, d\theta',$$

where $\cos\Theta$ has the meaning assigned above; r, θ, ϕ are the co-ordinates of the fixed point for which V_s is taken; and $f(r',\theta')$ is some function of the co-ordinates r',θ' of a point $(r'\theta'\phi')$ of the solid or surface respectively.

Our remarks will apply *mutatis mutandis* to both solid and surface, so that for brevity we restrict ourselves to the triple integral above set forth.

First suppose r to be less than r' throughout.

It may be given the form

$$\iiint_0^{2\pi} f(r', \theta') \sum_n r'^{\,2s-n-1} r^n \mathrm{P}_{n,s}(\cos\Theta)\, d\phi'\, d\theta'\, dr'. \quad (4)$$

For a point upon the z axis distant z from the origin, so that $\theta = 0$, we may suppose this expression equal to

$$\mathrm{M}\Sigma a_{n,s} z^n;$$

so that

$$\mathrm{M}a_{n,s} = \iiint_0^{2\pi} r'^{\,2s-n-1} f(r', \theta')\, \mathrm{P}_{n,s}(\mu')\, d\phi'\, d\theta'\, dr',$$
$$= 2\pi \iint r'^{\,2s-n-1} f(r', \theta')\, \mathrm{P}_{n,s}(\mu')\, d\theta'\, dr';$$

whence also

$$Ma_{n-2,\,s-1} = 2\pi \iint r'^{\,2s-n-1} f(r',\,\theta')\,P_{n-2,\,s-1}(\mu')\,d\theta'\,dr'$$

and in general

$$Ma_{n-2p,\,s-p} = 2\pi \iint r'^{\,2s-n-1} f(r',\,\theta')\,P_{n-2p,\,s-p}(\mu')\,d\theta'\,dr'.$$

Hence by (2) we see that the expression (4) for the potential may be written

$$M\sum_n \left\{ a_{n,\,s} P_{n,\,0}(\mu) + a_{n-2,\,s-1}\binom{2s}{2} P_{n,\,1}(\mu) + \ldots\ldots + a_{n-2p,\,s-p}\binom{2s}{2p} P_{n,\,p}(\mu) + \ldots\ldots \right\} r^n$$

or

$$M\sum_{n=0}^{n=\infty}\sum_{p}\binom{2s}{2p} a_{n-2p,\,s-p}\,P_{n,\,p}(\mu)\,r^n\,; \tag{5}$$

in which the upper limit of p is reached when $n-2p$ in either unity or zero; so far as the series does not terminate automatically before this point is reached by the vanishing of $\binom{2s}{2p}$.

5. Next suppose $r > r'$.

Then (3) may be written

$$\iiint_0^{2\pi} f(r',\,\theta')\sum_n r'^n r^{-(n-2s+1)} P_{n,\,s}(\cos\Theta)\,d\phi'\,d\theta'\,dr'. \tag{6}$$

For a point upon the z axis, where $\theta = 0$, we suppose this expression equal to

$$M\sum_n A_{n,\,s}\,z^{-(n-2s+1)},$$

and then we have

$$MA_{n,\,s} = \iiint_0^{2\pi} r'^n f(r',\,\theta')\,P_{n,\,s}(\mu')\,d\phi'\,d\theta'\,dr'$$

$$= 2\pi \iint r'^n f(r',\,\theta')\,P_{n,\,s}(\mu')\,d\theta'\,dr'$$

leading to the general relation

$$MA_{n,\,p} = 2\pi \iint r'^n f(r',\,\theta')\,P_{n,\,p}(\mu')\,d\theta'\,dr'$$

and therefore by (2)

$$\Sigma MA_{n,\,p}\binom{2s}{2p} P_{n-2p,\,s-p}(\mu) = 2\pi \iint r'^n f(r',\,\theta')\,P_{n,\,s}(\cos\Theta)\,d\theta'\,dr',$$

and the potential becomes

$$M\sum_n \left\{ A_{n,\,0} P_{n,\,s}(\mu) + A_{n,\,1}\binom{2s}{2} P_{n-2,\,s-1}(\mu) + \ldots\ldots + A_{n,\,p}\binom{2s}{2p} P_{n-2p,\,s-p}(\mu) + \ldots\ldots \right\} r^{-(n-2s+1)}$$

$$= Mr^{2s-1}\sum_{n=0}^{n=\infty}\sum_{0}^{p} A_{n,\,p}\binom{2s}{2p} P_{n-2p,\,s-p}(\mu)\,r^{-n}\,; \tag{7}$$

in which the upper limit of p is governed by the same considerations as in the expression (5).

6. Before formally enunciating the theorem that has been reached it may be verified, in its two cases, by applying ∇^2 to the expressions for the potential.

CASE 1.—$r < r'$.

Since

$$\nabla^2 (r^2 - 2rr' \cos \Theta + r'^2)^{s-1/2} = 2s(2s-1)(r^2 - 2rr' \cos \Theta + r'^2)^{s-3/2}$$

we have

$$\nabla^2 \iiint_0^{2\pi} f(r', \theta')(r^2 - 2rr' \cos \Theta + r'^2)^{s-1/2}\, d\phi'\, d\theta'\, dr'$$

$$= 2s(2s-1) \iiint_0^{2\pi} f(r', \theta')(r^2 - 2rr' \cos \Theta + r'^2)^{s-3/2}\, d\phi'\, d\theta'\, dr', \qquad (8)$$

and if (5) is the correct value of the potential the expression (8) may be deduced from (5) by writing therein $s-1$ for s and multiplying the result by $2s(2s-1)$.

The value of (8), thus obtained, is

$$2s(2s-1) \mathrm{M} \sum_{n=0}^{n=\infty} \sum_{p=0}^{p} \binom{2s-2}{2p} a_{n-2p,\, s-p-1} \mathrm{P}_{n,\, p}(\mu)\, r^n\,; \qquad (9)$$

moreover, since

$$\nabla^2 \mathrm{P}_{n,\, s} r^n = 2s(2s-1) \mathrm{P}_{n-2,\, s-1} r^{n-2},$$

the operation of ∇^2 upon (5) yields

$$\mathrm{M} \sum_{n=0}^{n=\infty} \sum^{p} \binom{2s}{2p} a_{n-2p,\, s-p}\, 2p(2p-1) \mathrm{P}_{n-2,\, p-1}(\mu)\, r^{n-2}.$$

This may be written

$$2s(2s-1) \mathrm{M} \sum_{n=0}^{n=\infty} \sum_{p=1}^{p} \binom{2s-2}{2p-2} a_{n-2p,\, s-p} \mathrm{P}_{n-2,\, p-1}(\mu)\, r^{n-2},$$

and, writing herein $n+2$ for n and $p+1$ for p, this is

$$2s(2s-1) \mathrm{M} \sum_{n=0}^{n=\infty} \sum_{p=0}^{p} \binom{2s-2}{2p} a_{n-2p,\, s-p-1} \mathrm{P}_{n,\, p}(\mu)\, r^n$$

agreeing with (9).

7. CASE 2.—$r > r'$.

Similarly operating with ∇^2 upon (3) we obtain (8) as before; and if (7) is the correct expression of the potential the value of (8) may be deduced from (7) by writing therein $s-1$ for s, and subsequently multiplying by $2s(2s-1)$.

The value of (8), thus obtained, is

$$2s(2s-1)\mathrm{M}r^{2s-3}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p}\binom{2s-2}{2p}\mathrm{A}_{n,p}\mathrm{P}_{n-2p,s-p-1}(\mu)r^{-n}; \tag{10}$$

moreover since

$$\nabla^2\mathrm{P}_{n,s}r^{2s-1-n} = 2s(2s-1)\mathrm{P}_{n,s-1}r^{2s-3-n},$$

the operation of ∇^2 upon (7) yields

$$\mathrm{M}r^{2s-3}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p}\binom{2s}{2p}\mathrm{A}_{n,p}(2s-2p)(2s-2p-1)\mathrm{P}_{n-2p,s-p-1}(\mu)r^{-n}$$

which is

$$2s(2s-1)\mathrm{M}r^{2s-3}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p}\binom{2s-2}{2p}\mathrm{A}_{n,p}\mathrm{P}_{n-2p,s-p-1}(\mu)r^{-n}$$

agreeing with (10).

8. The theorem being thus established, we may enunciate it in the following manner :—

"THEOREM.—The attracting body, of mass M and of uniform density, being a hollow shell whose exterior and interior bounding surfaces are surfaces of revolution about the axis of z; let the origin be taken within the interior bounding surface; let

$$\mathrm{R}_{z,0}, \qquad \mathrm{R}_{r,\theta}$$

be the distances from any points of the body to the points whose co-ordinates are z, 0, r, θ respectively.

Let
$$\Sigma\mathrm{R}_{z,0}^{2s-1}\,d\mathrm{M} = \mathrm{M}\Sigma a_{n,s}z^n,$$
or
$$= \mathrm{M}\Sigma\mathrm{A}_{n,s}z^{-(n-2s+1)},$$

the series which is convergent for the point $z, 0$ being chosen. Then the potential at the point r, θ is, when the series proceeding by positive powers of r is convergent, represented by

$$\Sigma\mathrm{R}_{r,\theta}^{2s-1}\,d\mathrm{M}$$
$$= \mathrm{M}\sum_n\left\{a_{n,s}\mathrm{P}_{n,0} + \binom{2s}{2}a_{n-2,s-1}\mathrm{P}_{n,1} + \ldots + \binom{2s}{2p}a_{n-2p,s-p}\mathrm{P}_{n,p} + \ldots\right\}r^n;$$

that is to say, it is obtained by substituting in the potential expression

$$\mathrm{M}\Sigma a_{n,s}z^n,$$

r for z, and

$$a_{n,s}\mathrm{P}_{n,0} + \binom{2s}{2}a_{n-2,s-1}\mathrm{P}_{n,1} + \ldots\ldots + \binom{2s}{2p}a_{n-2p,s-p}\mathrm{P}_{n,p} + \ldots$$

for

$$a_{n,s},$$

the upper limits of p in the series written being reached when $n-2p$ is

zero or unity, subject to the series not having already terminated by the vanishing of

$$\binom{2s}{2p}.$$

Also when the series proceeding by negative powers of r is convergent, the potential is represented by

$$\Sigma R_{r,\theta}^{2s-1}\, dM$$

$$= M\sum_n \left\{ A_{n,0}P_{n,s} + \binom{2s}{2} A_{n,1}P_{n-2,s-1} + \ldots \right.$$

$$\left. + \binom{2s}{2p} A_{n,p}P_{n-2p,s-p} + \ldots \right\} r^{-(n-2s+1)};$$

that is to say, it is obtained by substituting in the potential expression

$$M\Sigma A_{n,s} z^{-(n-2s+1)},$$

r for z, and

$$A_{n,0}\, P_{n,s} + \binom{2s}{2} A_{n,1} P_{n-2,s-1} + \ldots + \binom{2s}{2p} A_{n,p} P_{n-2p,s-p} + \ldots$$

for

$$A_{n,s}$$

the upper limit of p being determined as in the previous case."

9. It will be noticed that in the series

$$a_{n,s}\, P_{n,0} + \binom{2s}{2} a_{n-2,s-1} P_{n,1} + \ldots,$$

$$A_{n,0}\, P_{n,s} + \binom{2s}{2} A_{n,1} P_{n-2,s-1} + \ldots,$$

there is a remarkable correspondence.

If the former be denoted by

$$\chi\,(a, P)$$

the latter will be denoted by

$$\chi\,(P, A).$$

In the former series, every P has the first suffix n.

10. We propose now to give the detailed results for the case of a ring of attracting matter, of radius c and of indefinitely small cross-section.

For a ring of radius c, the potential at a point on the axis of z is

$$M\,(c^2+z^2)^{s-1/2}$$

and this may be expanded so that the numerical coefficients are functions $P_{n,s}$ with zero arguments. For convenience write

$$P_{n,s}(0) = P_{n,s}'.$$

Then if z is less than c the expanded form is

$$\mathrm{M}c^{2s-1}\left\{\mathrm{P}_{0,s}'+\mathrm{P}_{2,s}'\frac{z^2}{c^2}+\mathrm{P}_{4,s}'\frac{z^4}{c^4}+\ldots\ldots\right\}=\mathrm{M}c^{2s-1}\sum_n \mathrm{P}_{2n,s}'\left(\frac{z}{c}\right)^{2n}.$$

Hence $$a_{2n,s}=\mathrm{P}_{2n,s}'c^{-(2n-2s+1)}$$

and the formula gives for the potential at the point r, θ

$$\mathrm{M}c^{2s-1}\Bigg[\mathrm{P}_{0,s}'\mathrm{P}_{0,0}+\left\{\mathrm{P}_{2,s}'\mathrm{P}_{2,0}+\binom{2s}{2}\mathrm{P}_{0,s-1}'\mathrm{P}_{2,1}\right\}\frac{r^2}{c^2}$$

$$+\left\{\mathrm{P}_{4,s}'\mathrm{P}_{4,0}+\binom{2s}{2}\mathrm{P}_{2,s-1}'\mathrm{P}_{4,1}+\binom{2s}{4}\mathrm{P}_{0,s-2}'\mathrm{P}_{4,2}\right\}\frac{r^4}{c^4}$$

$$+\left\{\mathrm{P}_{6,s}'\mathrm{P}_{6,0}+\binom{2s}{2}\mathrm{P}_{4,s-1}'\mathrm{P}_{6,1}+\binom{2s}{4}\mathrm{P}_{2,s-2}'\mathrm{P}_{6,2}+\binom{2s}{6}\mathrm{P}_{0,s-3}'\mathrm{P}_{6,3}\right\}\frac{r^6}{c^6}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Bigg]$$

$$=\mathrm{M}c^{2s-1}\sum_n\left\{\mathrm{P}_{2n,s}'\mathrm{P}_{2n,0}+\binom{2s}{2}\mathrm{P}_{2n-2,s-1}'\mathrm{P}_{2n,1}+\ldots\ldots+\binom{2s}{2n}\mathrm{P}_{0,s-n}'\mathrm{P}_{2n,n}\right\}\left(\frac{r}{c}\right)^{2n}.$$

Also if z be greater than c, the expanded form is

$$\mathrm{M}z^{2s-1}\sum_n\mathrm{P}_{2n,s}'\left(\frac{c}{z}\right)^{2n}=\mathrm{M}z^{2s-1}\left\{\mathrm{P}_{0,s}'+\mathrm{P}_{2,s}'\left(\frac{c}{z}\right)^2+\mathrm{P}_{4,s}'\left(\frac{c}{z}\right)^4+\ldots\ldots\right\}.$$

Hence $$\mathrm{A}_{2n,s}=\mathrm{P}_{2n,s}'c^{2n}$$

and the formula gives for the potential at the point r, θ

$$\mathrm{M}r^{2s-1}\Bigg[\mathrm{P}_{0,0}'\mathrm{P}_{0,s}+\left\{\mathrm{P}_{2,0}'\mathrm{P}_{2,s}+\binom{2s}{2}\mathrm{P}_{2,1}'\mathrm{P}_{0,s-1}\right\}\frac{c^2}{r^2}$$

$$+\left\{\mathrm{P}_{4,0}'\mathrm{P}_{4,s}+\binom{2s}{2}\mathrm{P}_{4,1}'\mathrm{P}_{2,s-1}+\binom{2s}{4}\mathrm{P}_{4,2}'\mathrm{P}_{0,s-2}\right\}\frac{c^4}{r^4}$$

$$+\left\{\mathrm{P}_{6,0}'\mathrm{P}_{6,s}+\binom{2s}{2}\mathrm{P}_{6,1}'\mathrm{P}_{4,s-1}+\binom{2s}{4}\mathrm{P}_{6,2}'\mathrm{P}_{2,s-2}+\binom{2s}{6}\mathrm{P}_{6,3}'\mathrm{P}_{0,s-3}\right\}\frac{c^6}{r^6}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Bigg].$$

It will be noticed that the latter is obtainable from the former by interchanging accented aud unaccented P functions, and also r and c.

It will be clear that in the theory of attraction above considered the density need not be uniform throughout the mass. It will suffice if it is a function of r', θ', and not of ϕ'.